There may be no $\mathscr{I}$-ultrafilter for any F_σ ideal $\mathscr{I}$

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Introduction

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5. Given a filter $\mathcal{F}$ on ω, the **dual ideal** to $\mathcal{F}$, denoted by $\mathcal{F}^*$, is defined as the family of complements of elements from $\mathcal{F}$, that is,

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   In a similar fashion we define the **dual filter** to a given ideal $\mathscr{I}$, and write $\mathscr{I}^*$.
6. An ideal $\mathscr{I}$ on ω is a **p-ideal** if for any $\{A_n : n \in \omega\} \subseteq \mathscr{I}$ there is $B \in \mathscr{I}$ such that for all $n \in \omega$, $A_n \subseteq^* B$.

We assume that all our ideals are **tall**, that is, for all $A \in [\omega]^\omega$, there is an infinite $B \in \mathscr{I}$ such that $B \subseteq A$.

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3. Summable ideals: are defined by a function $f : \omega \to \mathbb{R}^+ \cup \{0\}$ such that $\Sigma_{n\in\omega} f(n) = \infty$, and $A \in \mathscr{I}_f$ if and only if $\Sigma_{n\in A} f(n) < \infty$. A typical example is given by the function $f(n) = 1/(n+1)$.

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All of these ideals have complexity F_σ.

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conv and Fin $\times$ Fin have complexity $F_{\sigma\delta\sigma}$, while nwd has complexity $F_{\sigma\delta}$.

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Let $\mathscr{I}$ be an ideal and $\mathcal{U}$ an ultrafilter, both of them on ω

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2. $\mathcal{U}$ is a weak $\mathscr{I}$-ultrafilter if for any finite to one function $f \in \omega^\omega$, there is $A \in \mathcal{U}$ such that $f[A] \in \mathscr{I}$.

Many combinatorial properties of ultrafilters on ω can be stated in terms of being an $\mathscr{I}$-ultrafilter for a suitable ideal $\mathscr{I}$, for example:

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5. $\mathcal{U}$ is a Hausdorff ultrafilter if and only if $\mathcal{U}$ is a $\mathcal{G}_{fc}$-ultrafilter.

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6. (S. Shelah) There is no ultrafilter with property M.

A natural question that raises by watching the previous examples is the following:

Question

Is there a Borel ideal $\mathscr{I}$ for which there is an $\mathscr{I}$-ultrafilter $\mathcal{U}$ or a weak $\mathscr{I}$-ultrafilter?

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Theorem(O. Guzmán González, M. Hrušák)

It is relatively consistent that for any $F_{\sigma\delta}$ ideal $\mathscr{I}$ generic existence of $\mathscr{I}$-ultrafilters does not hold, i. e., there is a filter with a small generating set($< 2^{\omega}$) which can not be extended to an $\mathscr{I}$-ultrafilter.

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Answer: Consistently no.

Theorem

It is relatively consistent with ZFC that for every F_σ ideal $\mathscr{I}$, $\mathscr{I}$-ultrafilters do not exist. Not even weak $\mathscr{I}$-ultrafilters.

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7. (M. Benedikt) Is there an ultrafilter with property M? (Originally answered by Shelah).

F_σ ideals

Definition

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6. $(\forall n \in \omega)(\varphi(\{n\}) < \infty)$.

Theorem(Mazur)

An ideal $\mathscr{I}$ is an F_σ ideal provided there is a lscsm φ such that $\mathscr{I} = Fin(\varphi) = \{A \in \mathcal{P}(\omega) : \varphi(A) < \infty\}$.

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9 For a tree $T \subseteq \omega^{<\omega}$ and a non-empty $F \subseteq \omega$, define $(T)_F = \bigcup_{n \in F} (T)_n$.

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8 Given a tree $T \subseteq \omega^{<\omega}$, we denote by $(T)_n$ the set of all nodes in T with length exactly n, that is $(T)_n = \{s \in T : |s| = n\}$.

9 For a tree $T \subseteq \omega^{<\omega}$ and a non-empty $F \subseteq \omega$, define $(T)_F = \bigcup_{n \in F} (T)_n$.

10 For a set $S \subseteq \omega^{<\omega}$, define the tree generated by S, denoted $gt(S)$, as follows:

$$gt(S) = \{s \in \omega^{<\omega} : (\exists r \in S)(s \subseteq r)\}$$

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Definition

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Definition

The Miller's forcing, denoted by PT, is the partial order whose members are all the superperfect trees, ordered by set inclusion, that is, given $S, T \in \mathrm{PT}$, $S \leq T$ if and only if $S \subseteq T$.

Notation

For $T \in \mathrm{PT}$, we denote by $st(T)$ the stem of condition T, which is the unique splitting node in T with minimal length.

In what follows we fix an F_σ ideal $\mathscr{I}$, and φ denotes a lscsm which defines the ideal $\mathscr{I}$, that is $\mathscr{I} = Fin(\varphi)$.

The forcing

Definición

Let $T \in \mathrm{PT}$ be a condition in the Miller's forcing and $m \in \omega$. We say that a node $s \in T$ is (T, φ, m)-good, if there is $F_s^T \in [\omega]^{<\omega}$ such that:

1. $|s| = \min(F_s^T)$.

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1. For any $m \in \omega$ and any $s \in T$, there is a (T, φ, m)-good node $t \in T$ extending the node s.

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2. For any node $s \in split(T)$, there is a (T, φ, m)-good node $r \in T$ for some $m \in \omega$, such that $s \in (T \restriction r)_{F_s^T}$.

The forcing

Definition

We define the forcing $\mathrm{PT}(\varphi)$ as the set of all superperfect trees which are φ-good, ordered by set inclusion.

Lemma 1 (Axiom A)

Lemma 1
The forcing $\mathrm{PT}(\varphi)$ has Axiom A, therefore $\mathrm{PT}(\varphi)$ is a proper forcing.

$\mathscr{I}$-ultrafilters destruction

Definition

Let $\dot{G}$ be a generic filter for $\mathrm{PT}(\varphi)$. We define the generic real $\dot{x}_{gen}$ as:

$$\dot{x}_{gen} = \bigcup \bigcap \dot{G}$$

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Let $\dot{x}_{gen}$ be the generic real added by $\dot{G}$. We define the function $\dot{f}_{gen}$ as:

$$\dot{f}_{gen}(n) = k \iff n \in (\dot{x}_{gen}(k-1), \dot{x}_{gen}(k)]$$

$\mathscr{I}$-ultrafilters destruction

Lemma 2

PT(φ) forces that for all $A \in [\omega]^\omega \cap V$, $\varphi(\dot{f}_{gen}[A]) = \infty$.

Continuous reading of names

Lemma 3

Let $\dot{x}$ be a PT(φ)-name for an infinite subset of ω, and let $T \in$ PT(φ) be a condition. There is $T' \leq T$ such that:

1. $F^{T'}_{st(T')} = F^{T}_{st(T)}$.

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2. For each $f \in (T')_{F^{T'}_{st(T')}}$ there is a set $X_f \subseteq \omega$ such that for all $n \in \omega$ and for all but finitely many $k \in succ_{T'}(f)$:

$$T' \upharpoonright f^\frown k \Vdash \text{“}\dot{x} \cap n = X_f \cap n\text{”}$$

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Proof: induction over the size of $F^{T}_{st(T)}$.

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1. $\varphi(F^{T'}_{st(T')}) \geq \varphi(F^{T}_{st(T)})/2$.

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    1. For all $f \in (T')_{F^{T'}_{st(T')}}$, $X_f \in \mathcal{U}$.
    2. For all $f \in (T')_{F^{T'}_{st(T')}}$, $\omega \setminus X_f \in \mathcal{U}$.

Continuous reading of names

Lemma 5

Let $T \in \mathrm{PT}(\varphi)$ and $c : split(T) \to 2$ is a coloring, then there is a condition $T' \leq T$ such that $c \restriction split(T')$ is constant.

Continuous reading of names

Lemma 6

Let $\mathcal{U}$ be an ultrafilter, $T \in \mathrm{PT}(\varphi)$ be a condition, $\dot{x}$ be a $\mathrm{PT}(\varphi)$-name. Then there is $T' \leq T$ such that for all $s \in split(T')$, there is $X_s \subseteq \omega$ satisfying the following two conditions:

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   1. For all $s \in split(T')$, $X_s \in \mathcal{U}$.
   2. For all $s \in split(T')$, $\omega \setminus X_s \in \mathcal{U}$.
2. For all $s \in split(T')$, for all $n \in \omega$ and for all but finitely many $k \in succ_{T'}(s)$:

$$T' \restriction s^\frown k \Vdash \text{“}\dot{x} \cap n = X_s \cap n\text{”}$$

Continuous reading of names

Lemma 6(version 2)

Let $\dot{x}$ be a $\mathrm{PT}(\varphi)$-name for a function from ω to ω, and $T \in \mathrm{PT}(\varphi)$ be a condition which forces $\dot{x}$ to be bounded by $g \in \omega^\omega$. Then there are $T' \leq T$ and $S \subseteq split(T')$ which gives φ-block structure to T', such that for all $s \in S$:

> For each $r \in (T')_{F_s^{T'}}$ there is a function $f_r \in \omega^\omega$ such that for all $n \in \omega$, for all but finitely many $k \in succ_{T'}(r)$:
>
> $$T' \restriction r^\frown k \Vdash \text{“}\dot{x} \restriction (|r| + n) = f_r \restriction (|r| + n)\text{”}$$

Laver Property

Proposition 7

The forcing $\mathrm{PT}(\varphi)$ has the Laver property.

P-points preservation

Proposition 8

The forcing $\mathrm{PT}(\varphi)$ preserves p-points.

Near Coherence of Filters principle

For $\mathcal{U}$ an ultrafilter on ω and $f \in \omega^\omega$, $f(\mathcal{U}) = \{A \in \mathcal{P}(\omega) : f^{-1}[A] \in \mathcal{U}\}$.

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Definition(NCF, A. Blass)

Given two ultrafilters on ω, $\mathcal{U}$ and $\mathcal{V}$, there is a finite to one function $f \in \omega^\omega$ such that $f(\mathcal{U}) = f(\mathcal{V})$.

Near Coherence of Filters

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The following statements are consequences of the Near Coherence of Filters principle:

1. There are ultrafilters generated by less than $\mathfrak{d}$ sets.

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The following statements are consequences of the Near Coherence of Filters principle:

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3. p-points are dense in the Rudin-Blass ordering.
4. There are no q-points.

Near Coherence of Filters

Lemma 10

Let $\mathcal{U}$ and $\mathcal{V}$ be two ultrafilters on ω. Then $\mathrm{PT}(\varphi) \Vdash$ "$\dot{f}_{gen}(\mathcal{V}) = \dot{f}_{gen}(\mathcal{U})$". Moreover, for each ultrafilter $\mathcal{U}$ on ω, $\mathrm{PT}(\varphi) \Vdash$ "$\dot{f}_{gen}(\mathcal{U})$ is a p-point".

Near Coherence of Filters

Theorem(A. Blass, S. Shelah)

Let $\mathbb{P}_\alpha = \langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ be a countable support iteration of proper forcings such that for all $\beta < \alpha$, $\mathbb{P}_\beta$ forces that $\dot{\mathbb{Q}}_\beta$ preserves *p*-points. Then $\mathbb{P}_\alpha$ is proper and preserves *p*-points.

The model

Theorem

Let $\mathrm{P}_{\omega_2} = \langle \mathrm{P}_\beta, \dot{\mathrm{Q}}_\beta : \beta < \omega_2 \rangle$ be a countable support iteration of proper forcings such that for any $\alpha < \omega_2$, P_α forces that $\dot{\mathrm{Q}}_\alpha$ is of the form $\mathrm{PT}(\dot{\varphi})$, and for any lscsm $\dot{\varphi}$ which appears in the intermediate steps, $\mathrm{PT}(\dot{\varphi})$ appears cofinally often. Then P_{ω_2} forces that for any F_σ ideal $\mathscr{I}$, there is no $\mathscr{I}$-ultrafilter. In particular, there is no Hausdorff ultrafilter in the resulting model.

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In the previous model *NCF* is true and the following holds true that for any F_σ ideal $\mathscr{I}$:

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In the previous model *NCF* is true and the following holds true that for any F_σ ideal $\mathscr{I}$:

$$(\forall X \subseteq [\omega]^\omega)(|X| \leq \aleph_1 \Rightarrow (\exists f \in \mathrm{FtO})(f[X] \in \mathscr{I}^+))$$

Question

Is there an $F_{\sigma\delta}$ ideal $\mathscr{I}$, in ZFC, such that $\mathscr{I}$-ultrafilters exist?

Thank you very much for you attention!